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Ashish Prakash, Richa Agarwal and Rakhi Trivedi

Duality in Mathematical Fractional Programming Optimality under Convexity

Ashish Prakash[1, a)], Richa Agarwal[2, b)] and Rakhi Trivedi[3,c)]

[1]ABES Engineering College, Campus-1, NH-24, Ghaziabad, UP, India
[2] KIET Group of Institutions, Ghaziabad, UP, India
[3] IILM College of Engineering and Technology, Greater Noida, UP, India

[a)]Corresponding author:apgarg@abes.ac.in
[b)] richaagarwal796@gmail.com
[c)] trivedirakhi80@gmail.com

Abstract- In this paper, weir-type dual with idea of effectiveness is used to utter duality theorems, under generalized (F, ρ) convexity assumption for multi objective fractional programming problem. This work is an extension of the outcome of Mustafa in the environment of multi-objective fractional programming using generalized (F, ρ) convexity assumption.

INTRODUCTION

Egudo derived specialtheorems of duality for multi objective fractional programming [4] (MFP) problem, using the concept of effectiveness tied with universal ρ-convexity [10]. After that Mukherjee gave the extension of the outcomes of Egudo in the perspective of multi objective fractional [13] programming, using parametric approach. For multi-objective nonlinear programming [2] problem, inequality and equality constraints are involved in this, Preda used the idea of effectiveness to state the results of duality under universal (F,ρ)-convexity. Yong extended and unified the results of Mukherjee. He established specialtheorems of duality for multi objective fractional programming problem [1], via the method of effectiveness fixed with special universal convex assumption.

Consider the following fractional programming issue with many objectives:

(MFP) Minimize$\left(\frac{f_1(x)}{g_1(x)},\frac{f_2(x)}{g_2(x)},\ldots\ldots\ldots\ldots,\frac{f_p(x)}{g_p(x)}\right)$ Subject to $h(x) \leq 0$ --------------------(1)

Where the functions which satisfy differentiability [11] and$g_i > 0$ are given by $f_i: R^n \to R, g_i: R^n \to R \; for \; i = 1,2,3, \ldots, p \; and \; h: R^n \to R^m$

I. The following definition will be used in establishing results of this paper

DEFINITION 2.1:

x^0 is a feasible solution for (MFP) which is suitable result for (MFP) if for (MFP)feasible solution does not exist like x* such that $\frac{f_i(x^*)}{g_i(x^*)} < \frac{f_i(x^0)}{g_i(x^0)}$ for some $i = 1,2,3, \ldots, p$

And $\frac{f_j(x^*)}{g_j(x^*)} \leq \frac{f_j(x^0)}{g_j(x^0)}$ *for all* $j = 1,2,3, \ldots, p$

Consider Weir type dual as follows for (MFP):

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(MFD) $Maximize \left\{\frac{f_1(y)}{g_1(y)} \frac{f_2(y)}{g_2(y)} \ldots\ldots\ldots\ldots \frac{f_p(y)}{g_p(y)}\right\}$

$y_i t_i \sigma$

Subject to $\sum_{i=1}^{p} \tau_i \nabla \frac{f_i(y)}{g_i(y)} + \nabla \sigma^t \square(y) = 0$ ---------------(2)

$\sigma^t \square(y) \geq 0$ ---------------(3)

$\sigma \geq 0, \tau_i \geq 0, \quad i = 1, \ldots\ldots\ldots, p, \quad \sum_{i=1}^{p} \tau_i = 1$ ----------------(4)

The following outcomes will be necessary in order to prove strong duality [12].

Lemma-2.1x* is an appropriate solution for (MFP) iffx* solves (MFP$_k(\delta^0)$), for each

(MFP$_k(\delta^0)$) Minimize $\frac{f_k(X)}{g_k(X)}$

Subject to$\frac{f_j(X)}{g_j(X)} \leq \delta_j^0$for all j≠k

h(x)≤0 where $\delta_1^0 = \frac{f_j(X^*)}{g_j(X^*)}$

If $g_i(x) > 0$ for every $j = 1,2,3, \ldots, p$ so (MFP$_k(\delta^0)$) can be write as

(MFP$_k(\delta^0)$)Minimize $\frac{f_k(X)}{g_k(X)}$

Subject to $f_j(x) - \delta_j^0 g_j(x) \leq 0$ for all $j \neq k$ -------------(5)

$h(x) \leq 0$

Lemma-2.2**:** If in (MFP) assume$g_j(x) > 0$ when$j = 1,2,3, \ldots, p$ then x^* solves (MFP'$_k(\delta^0)$) for each $k = 1,2,3, \ldots, p$ iffx^* is a suitable solution of (MFP).

Lemmas 1 & 2 given by Egudo are similar to Lemma 2.1 &2.2.

II. Duality theorems [3] with Generalized [9] F-Convexity:

Under this section, weak and strong duality results between (MFP) & (MFD) are established, using generalized F-convexity assumption on the functions involved.

Theorem 3.1.(Weak duality) If x^0 is feasible for (MFP) and (y^0, τ, σ) is feasible for (MFD), f_i,is nonnegative and F-convex, g_i, is positive & F-concave for everyi=1…, p and σ^t is F-quasi-convex for y^0. If also either of the given hypotheses holds,

(a) $\tau_i > 0$for all i=1…......, p.

(b) $\sum_{i=1}^{p} \tau_i \frac{f_i(.)}{g_i(.)x}$is strictly F-pseudo [7] convex at y^0,

then the following never hold,

$\frac{f_j(X^0)}{g_j(X^0)} \leq \frac{f_j(Y^0)}{g_j(Y^0)}$ for all j=1,…..........,p ------- (6)

and

$\frac{f_i(X^0)}{g_i(X^0)} \leq \frac{f_i(Y^0)}{g_i(Y^0)}$ for some i=1,…......,p ------------- (7)

Proof: Since x^0 is feasible to (MFP) and (y^0,τ, σ) is feasible to (MFD), then we have

$\sigma^t h(x^0) \leq \sigma' h(y^0)$ $Since\ \sigma^t \square\ is\ F\text{-}quasi\ convex$

⇒ F (x^0, y^0, $\nabla \sigma^t$h(y^0))≤0 ---------------(8)

equations (2) and (8) imply $F\left(x^0, y^0, \sum_{i=1}^{p} \tau_i \nabla \frac{f_i(y^0)}{g_i(y^0)}\right) \geq 0$ ---------------(9)

Suppose if the results (6) and (7) of the theorem hold in contrary. From hypothesis (a) it follows that

$\tau_j \frac{f_j(X^0)}{g_j(X^0)} \geq \tau_j \frac{f_j(y^0)}{g_j(y^0)}$ for all j = l,….,p -------------(10)

and

$\tau_i \frac{f_i(X^0)}{g_i(X^0)} \geq \tau_i \frac{f_i(y^0)}{g_i(y^0)}$ for all i = l,….,p --------------(11)

If $f_j(x^0) \geq 0$ and $g_j(x^0) > 0$ for each j∈P={l,…,p}, it follows that

$T_j \frac{f_j(X^0)}{g_j(X^0)}, j = 1, \ldots\ldots, p$ are F-pseudo convex at y^0

Hence (10) and (11) imply

$$F\left(x^0, y^0, \sum_{i=1}^{p} \tau_i \nabla \frac{f_i(y^0)}{g_i(y^0)}\right) < 0$$

which contradicts (9)

Now on applying $\tau_j \geq 0$, i=1,….,.. p (since τ is feasible for (MFD)) in (6) and (7) we obtain

$$\sum_{i=1}^{p} \tau_i \frac{f_i(X^0)}{g_i(X^0)} \leq \sum_{i=1}^{p} \tau_i \frac{f_i(y^0)}{g_i(y^0)} \quad \text{--------------(12)}$$

Hypothesis (b) and (12) now imply

$$F\left(x^0, y^0, \sum_{i=1}^{p} \tau_i \nabla \frac{f_i(y^0)}{g_i(y^0)}\right) < 0$$

which again contradicts (9).

Hence weak duality [3] follows.

Theorem 3.2: (Strong duality) Suppose x^0 be a suitable result to (MFP) & suppose it also satisfy constraint qualification to $(MFP_k(\delta^0))$ for at least one k=1,..,p. So $\exists \tau^0 \in R^p$ with $\sigma^0 \in R^m$ so that (x^0, τ^0, σ^0) is a feasible result to (MFD). If weak duality between (MFP) and (MFD) holds, then (x^0, τ^0, σ^0) is an proficient solution [6] for (MFD).

Proof: From Lemma 2.1, x^0 solves $(MFP_k(\delta°))$ for all k∈P=(1,…,p) and efficient [5] for (MFP). Then ∃ a k∈P so that x^0 satisfies constraints qualification for $(MFP_k(\delta^0))$, by hypothesis.

Now from Kuhn-Tucker necessary condition for $\tau_i \geq 0$ for all i≠k and $\sigma \geq 0$ such that

$$\nabla\left[\frac{f_k(X^0)}{g_k(X^0)} + \sum_{i \neq k} \tau_i \frac{f_i(X^0)}{g_i(X^0)} + \sigma^t \square(X^0)\right] = 0 \text{--------------(13)}$$

and $\sigma^t \square(x^0) = 0$--------------(14)

If we put $\tau_k^0 = \frac{1}{1+\sum_{i \neq k} \tau_i} > 0$ and divide all terms in (13) and (14) by, $1 + \sum_{i \neq k} \tau_i$ we find

$$\tau_k^0 = \frac{\tau_j}{1 + \sum_{i \neq k} \tau_i} \geq 0 \quad \text{and} \quad \sigma_1^0 = \frac{\sigma}{1 + \sum_{i \neq k} \tau_i} \geq 0$$

Here it is concluded that (x^0, τ^0, σ^0) is also feasible for (MFD) and weak duality between (MFP) and (MFD) holds. It shows (x^0, τ^0, σ^0) is an effective result for (MFD).

I. Duality Theorems with Generalized (F, ρ) – Convexity:

Again, weak and strong duality results [8] were established by us, but for universal (F,$\boldsymbol{\rho}$)-convexity assumption on the functions involved in the (MFP) and (MFD).

Theorem 4.1: (Weak duality): If with each feasible x° to (MFP) and each feasible $(y^0,\tau,\sigma)$to (MFD), f_i is nonnegative and (F, $\boldsymbol{\rho}^i$)-convex g_i is positive and F-concave for each i=1,…,p and that σ^th is (F, $\boldsymbol{\rho}^0$)-quasi-convex at y^0. If also either of the given hypotheses holds,

(a) τ_i>0 forall i=1,**…,**p.

(b) $\sum_{i=1}^{p}\tau_i \frac{f_i(.)}{g_i(.)}$ is strictly (F, $\boldsymbol{\rho}^1$) pseudo convex at y^0

and if$\boldsymbol{\rho}^0$+$\boldsymbol{\rho}^1$> 0,

then the following cannot hold,

$$\frac{f_j(X^0)}{g_j(X^0)} \le \frac{f_j(y^0)}{g_j(y^0)} \qquad \text{for all j=1,…....p,} \qquad \text{…................(15)}$$

and

$$\frac{f_i(X^0)}{g_i(X^0)} < \frac{f_i(y^0)}{g_i(y^0)} \qquad \text{for some i = l,…,p} \qquad \text{……………(16)}$$

Proof: For the feasibility of x^0 and (y^0, τ,σ) in (MFP) and (MFD) , we have

$\sigma^t\square(x^0) \le \sigma^t\square(Y^0)$ Sinceσ^Th is (F,ρ^0) quasi convex at y^0

$\Rightarrow F(x^0,y^0,\nabla\sigma^t\square(y^0)) \le -\rho^0 d^2(x^0,y^0)$ -- …................(17)

Since $\rho^0+\rho^1>0$ and from (2) and (17) we have

$F\left(x^0,y^0,\sum_{i=1}^{p}\tau_i\nabla\frac{f_i(y^0)}{g_i(y^0)}\right) > -\rho^1 d^2(x^0,y^0)$…...................(18)

Now suppose the contrary, that the results (15) and (16) of the theorem hold. From hypothesis (a) it follows that

$$\tau_j\frac{f_j(x^0)}{g_j(x^0)} \le \tau_j\frac{f_j(y^0)}{g_j(y^0)} \qquad \text{for all j=1,…........,p} \qquad \text{…................(19)}$$

and

$$\tau_i\frac{f_i(x^0)}{g_i(x^0)} \le \tau_i\frac{f_i(y^0)}{g_i(y^0)} \qquad \text{for some i=1,…..,p} \qquad \text{…................(20)}$$

If $f_j(x^0)\ge 0$ and $g_j(x^0)>0$ for all j∈P it follows that $\sum_i^p \tau_j\frac{f_j(x^0)}{g_j(x^0)}$ is (F, ρ^1)-pseudo convex at y_0.

Hence (19) and (20) imply$F\left(x^0,y^0\sum_{i=1}^{p}\tau_i\nabla\frac{f_i(y^0)}{g_i(y^0)}\right) < -\rho^1 d^2(x^0,y^0)$ which contradicts (18)

Now using $\tau_i \ge 0$, i =1,**…,p** (since$\tau$is feasible to (MFD)) in (15) & (16), we find

$$\sum_{i=1}^{p}\tau_i\frac{f_i(x^0)}{g_i(x^0)} \le \sum_{i=1}^{p}\tau_i\frac{f_i(y^0)}{g_i(y^0)} \qquad \text{…................(21)}$$

Hypothesis (b) and (21) now imply

$$F\left(x^0,y^0\sum_{i=1}^{p}\tau_i\nabla\frac{f_i(y^0)}{g_i(y^0)}\right) < -\rho^1 d^2(x^0,y^0)$$

it againcontradicts (18). Hence weak duality follows.

Theorem 4.2: (Strong duality) If x^0 be an exact solution to (MFP) and suppose it also satisfy constraint qualification to ($MFP_k(\delta^0)$) for at least one k=1,….,p. Then $\exists\tau^0 \in R^p$ and $\sigma^0 \in R^m$ so that (x^0,τ^0,σ^0) is a feasible solution to (MFD). If weak duality theorem between (MFD) and (MFP) holds, then (x^0,τ^0,σ^0) is exact solution for (MFD).
Proof is similar to that of Theorem 3.2.

CONCLUSION

The goal of this paper is to demonstrate that for mathematical programming optimality and duality theorems involving standard duals with convexity assumptions, they can be obtained in a variety of ways: first, in modified convexity, then in complex spaces, third, in a modified dual, and finally, in the setting of non-smooth programmers.

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